Ablowitz-Ladik system with discrete potential. I. Extended resolvent

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Abstract

Ablowitz-Ladik linear system with range of potential equal to $\{0,1\}$ is considered. The extended resolvent operator of this system is constructed and the singularities of this operator are analyzed in detail.

1 Introduction

Our aim in this article is to study the spectral theory of the matrix operator $L(w)$,

$$\begin{aligned} L_{m,n}(w) &= \delta_{m,n-1} - \begin{pmatrix} w & r_n \\ s_n & 1/w \end{pmatrix} \delta_{m,n}, \\ m,n &\in \mathbb{Z}, \qquad w\in\mathbb{C}, \end{aligned} \tag{1.1}$$

every element of which is a 2×2 matrix, $\delta_{m,n}$ is the Kronecker symbol and we omitted a 2×2 unit matrix factor in the term $\delta_{m,n-1}$. Our attention is concentrated to the case where values of both potentials, r_n and s_n, are equal to 0 and 1:

$$r_n, s_n \in \{0,1\}, \qquad n\in\mathbb{Z}. \tag{1.2}$$

Moreover, we consider here the case of potentials with finite support, i.e., for every given potential there exist finite k and K, $k\le K$, $k,K\in\mathbb{Z}$—lower and upper borders of the support—such that

$$r_n = s_n = 0, \qquad n\le k-1, \qquad n\ge K+1. \tag{1.3}$$

The corresponding linear problem,

$$L(w)\Phi = 0, \tag{1.4}$$

is the Ablowitz–Ladik problem [1,2] which is known to be a discretized version of the Zakharov–Shabat linear problem. And like the latter the Ablowitz–Ladik problem is associated to a variety of differential–difference integrable equations, such as discrete mKdV equation, difference KdV, Toda chain, etc., [3]. Problem (1.4) describes also discrete systems with nonanalytic dispersion relations [4].

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The Ablowitz–Ladik problem is also known [5,6] to be associated to difference–difference nonlinear equations, that are related to some class of cellular automata, i.e., dynamical systems in a discrete space–time with values belonging to some finite field, say, $\mathbb{F}_2$. Cellular automata attract great interest in the literature because of the wide range of their applications in different sciences, from physics to biology, from chemistry to social sciences. Detailed references for these applications can be found in [7–11]. These automata are also subject to intensive mathematical study, see for example [12–22]. It is just this kind of applications of problem (1.4) that motivated our specific choice of condition (1.2) on potential.

The problem of the investigation of (1.4) by means of the inverse scattering transform, as it was performed in [3], becomes obvious if we write down this equation explicitly:

$$\Phi_{n+1} = \begin{pmatrix} w & r_n \\ s_n & 1/w \end{pmatrix} \Phi_n, \qquad n = 0, \pm 1, \pm 2, \dots . \tag{1.5}$$

In the standard approach to the study of the spectral problems, the main objects of the theory—the Jost solutions—are determined by their asymptotics at $n \to +\infty$ and $n \to -\infty$. A solution given by its asymptotics at $n \to -\infty$ is swept from the left by (1.5). But in order to construct the Jost solution given by its asymptotics at $n \to +\infty$, one has to invert the matrix in the r.h.s. of (1.5). The determinant of this matrix is equal to $1 - r_n s_n$, so in the standard approach the condition $r_n s_n \neq 1$ must be fulfilled. In the case where the potential satisfies (1.2) this means that for every n either r_n or s_n must be equal to zero [6]. Such condition drastically restricts the class of potentials of the type (1.2), so our aim in this and forthcoming publications is to elaborate an extension of the inverse scattering transform method to the case where both r_n and s_n can be equal to 1. Let us also emphasize that, imposing condition (1.2) on the potentials, we do not use here the condition $r_n, s_n \in \mathbb{F}_2$. As was speculated in [23] the problem of the integrability of the cellular automata or, more precisely, the problem of existence of the Lax representations must be solved in terms of the exact equalities, and not in terms of equalities on some finite field.

The fact that some matrix(–matrix) operator L is analogous to a differential one is reflected in the property that matrix elements $L_{m,n}$ are different from zero only for uniformly bounded values of $|m - n|$. In the case of (1.1) we have $1 \geq m - n \geq 0$. Consequently, we can apply the resolvent approach [24], [25] to investigation of the Ablowitz–Ladik problem. The preliminary results of our investigation were published in [26].

The resolvent approach is based on the following extension of the operator $L(w)$:

$$L_{m,n}(w, h) = h^{n-m} L_{m,n}(w), \tag{1.6}$$

where h is a real non-negative parameter. In particular for the operator (1.1) we have

$$L_{m,n}(w, h) = h\delta_{m,n-1} - u_n(w)\delta_{m,n}, \tag{1.7}$$

where we introduced

$$u_n(w) = \begin{pmatrix} w & r_n \\ s_n & 1/w \end{pmatrix} \equiv w^{\sigma} + \begin{pmatrix} 0 & r_n \\ s_n & 0 \end{pmatrix}, \qquad n \in \mathbb{Z}, \tag{1.8}$$

and σ is the Pauli matrix σ_3,

$$\sigma = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we have some infinite matrix–matrix operator $A_{m,n}(h)$ depending on a parameter h we can associate to it the Laurent series

$$A(\zeta,\zeta',h)=\sum_{m,n=-\infty}^{+\infty}\zeta^{-m}\zeta'^{n}A_{m,n}(h),\qquad \zeta,\zeta'\in\mathbb{C},\qquad |\zeta|=|\zeta'|=1. \tag{1.9}$$

In what follows we consider matrices $A_{m,n}(h)$ such that the series (1.9) are convergent in the sense of Schwartz distributions in ζ, ζ' ($|\zeta|=|\zeta'|=1$) and h ($h\geq 0$). The elements $A_{m,n}(h)$ are reconstructed by means of the formula

$$A_{m,n}(h)=\oint\limits_{|\zeta|=1}\frac{d\zeta\,\zeta^{m-1}}{2\pi i}\oint\limits_{|\zeta'|=1}\frac{d\zeta'\,\zeta'^{-n-1}}{2\pi i}A(\zeta,\zeta',h). \tag{1.10}$$

In order to explain the meaning of the extension (1.6) let us introduce the function (distribution)

$$A(\zeta,z)=A(\zeta e^{i\arg z},e^{i\arg z},|z|) \tag{1.11}$$

where $\zeta,z\in\mathbb{C}$, $|\zeta|=1$; by (1.6) and (1.9)

$$A(\zeta,z)=\sum_{m,n}z^{n-m}\zeta^{-m}A_{m,n}. \tag{1.12}$$

Then the above mentioned similarity of matrix and differential operators means that $A(\zeta,z)$ depends on z and z^{-1} polynomially. Let us mention that if we have two objects of this kind, A and B, their product (composition) is defined as follows:

$$(AB)_{m,n}(h)=\sum_{l=-\infty}^{+\infty}A_{m,l}(h)B_{l,n}(h), \tag{1.13}$$

$$(AB)(\zeta,\zeta'',h)=\oint\limits_{|\zeta'|=1}\frac{d\zeta'\,\zeta'^{-n-1}}{2\pi i}A(\zeta,\zeta',h)B(\zeta',\zeta'',h), \tag{1.14}$$

$$(AB)(\zeta,z)=\oint\limits_{|\zeta'|=1}\frac{d\zeta'}{2\pi i\zeta'}A(\zeta\overline{\zeta'},z\zeta')B(\zeta',z), \tag{1.15}$$

where the left hand sides of these equations are related through of (1.9)–(1.12). The main object of our investigation is inverse $M(w)$ of the operator $L(w)$ extended by (1.6),

$$L(w)M(w)=I,\qquad M(w)L(w)=I. \tag{1.16}$$

In matrix notations the first equality thanks to (1.7) has the form

$$hM_{m+1,n}(w,h)=\delta_{m,n}+u_m(w)M_{m,n}(w,h). \tag{1.17}$$

In order to define this inversion in a unique way we introduce

Definition 1. A solution $M(w)$ of (1.16) is called extended resolvent of the operator $L(w)$ if $M(w,\zeta,\zeta',h)$ is a Schwartz distribution with respect to ζ and ζ' and a sectionally continuous function of h, $h\geq 0$.

Let us first consider the case of zero potential, i.e., $r_n\equiv s_n\equiv 0$. Then the resolvent which we denote by $M_0(w)$ obeys the following equation

$$hM_{0,m+1,n}(w,h)=\delta_{m,n}+w^{\sigma}M_{0,m,n}(w,h). \tag{1.18}$$

It is convenient to rewrite this equation using representation (1.12):

$$(\zeta z - w^{\sigma})\, M_0(w,\zeta,z) = \delta_c(\zeta - 1), \tag{1.19}$$

where we introduced the δ-function on $|\zeta| = 1$,

$$\delta_c(\zeta - 1) = \sum_{n=-\infty}^{\infty} \zeta^n, \tag{1.20}$$

so that

$$\oint\limits_{|\zeta|=1} \frac{d\zeta}{2\pi i}\delta_c(\zeta - 1) f(\zeta) = f(1) \tag{1.21}$$

for an arbitrary test function $f(\zeta)$ on the contour. Then

$$M_0(w,\zeta;z) = (\zeta z - w^{\sigma})^{-1}\delta_c(\zeta - 1), \tag{1.22}$$

so that by (1.11)

$$M_0(w,\zeta,\zeta',h) = (\zeta h - w^{\sigma})^{-1}\delta_c(\zeta/\zeta' - 1), \tag{1.23}$$

or by (1.10)

$$\begin{aligned} M_{0,m,n}(w,h) = h^{n-m} w^{\sigma(m-n-1)} \Big\{ & \theta(h - |w^{\sigma}|)\theta(m \geq n+1) - \\ & -\theta(|w^{\sigma}| - h)\theta(n \geq m) \Big\}, \end{aligned} \tag{1.24}$$

where we introduced the matrices

$$\begin{aligned} \theta(h - |w^{\sigma}|) &= \begin{pmatrix} \theta(h - |w|) & 0 \\ 0 & \theta(h - 1/|w|) \end{pmatrix}, \\ \theta(|w^{\sigma}| - h) &= \begin{pmatrix} \theta(|w| - h) & 0 \\ 0 & \theta(1/|w| - h) \end{pmatrix}. \end{aligned} \tag{1.25}$$

Here we have to make some comments. First, by (1.6), all expressions $h^{m-n}L_{m,n}(w,h)$ are independent on h and equal to $L_{m,n}(w)$, see (1.1). On the contrary, $h^{m-n}M_{0,m,n}(w,h)$ essentially depends on h and it is just this dependence that guaranties that $M_0(w,\zeta,\zeta',h)$ exists as a distribution in ζ, ζ'. Second, any solution of the homogeneous equation $L_0(w)M_0(w) = 0$ is proportional to $\delta(h - |w^{\sigma}|)$, where the matrix δ-function is defined in analogy with (1.25). Thus we see that the condition set on $M_0(w,\zeta,\zeta',h)$ in Definition 1 to be a sectionally continuous function of h enables us to define the resolvent $M_0(w)$ uniquely. In what follows we consider the case of a nontrivial potential satisfying conditions (1.2) and (1.3).

2 Extended resolvent of the regularized operator

The specific problem connected with equation (1.17) is, as was mentioned above in the discussion of Eq. (1.4), that if $r_n = s_n = 1$ the matrix $u_n(w)$ is not invertible. Thus, first of all we have to introduce some regularization of $u_n(w)$, say,

$$\begin{aligned} u_n(w) \to u_n(w,\lambda) &= \begin{pmatrix} (\lambda r_n s_n + 1)w & r_n \\ s_n & 1/w \end{pmatrix} = \\ &= \lambda(1 - \det u_n(w))\frac{1+\sigma}{2} + u_n(w). \end{aligned} \tag{2.1}$$

This substitution regularizes singular only u_n (i.e., such that $\det u_n = 0$), leaving all other u_n untouched. Indeed, by (1.2) $\det u_n$ equals either 0 ot 1. Then

$$\det u_n(w,\lambda) = \begin{cases} 1, & \det u_n(w) = 1, \\ \lambda, & \det u_n(w) = 0. \end{cases} \tag{2.2}$$

Thus we start with the regularized operator

$$L_{m,n}(w,\lambda,h) = h\delta_{m,n-1} - u_m(w,\lambda)\delta_{m,n}$$

(cf. (2.40)), i.e., by (2.1)

$$L(w,\lambda) = L(w) - \lambda D, \tag{2.3}$$

where we introduced the diagonal operator

$$D_{m,n} = \frac{1+\sigma}{2}(1-\det u_n)\delta_{m,n}. \tag{2.4}$$

Correspondingly, we denote the extended resolvent of the regularized operator as $M(w,\lambda)$. It obeys equations (cf. (1.16))

$$L(w,\lambda)M(w,\lambda) = I, \qquad M(w,\lambda)L(w,\lambda) = I, \tag{2.5}$$

that by means of (2.3) can be written in the form

$$[L(w) - \lambda D]M(w,\lambda) = I, \tag{2.6}$$
$$M(w,\lambda)[L(w) - \lambda D] = I. \tag{2.7}$$

Properties of $M(w,\lambda)$ in the limit $\lambda \to 0$ are studied in the next section.

Let for simplicity write

$$\widetilde{M}_{m,n} = h^{m-n} M_{m,n}(w,\lambda,h), \tag{2.8}$$

i.e. we omit for a while dependencies on w, λ, and h. Then Eq. (2.6) takes the form

$$\widetilde{M}_{m+1,n} = \delta_{m,n} + u_m \widetilde{M}_{m,n}, \tag{2.9}$$

where dependence of u_m on w and λ is also omitted. It is easy to check that for any $m \geq m'$ we have from (2.9)

$$\widetilde{M}_{m,n} = \theta(m \geq n+1)\theta(n \geq m') \overleftarrow{\prod_{l=n+1}^{m-1}} u_l + \left(\overleftarrow{\prod_{l=m'}^{m-1}} u_l \right) \widetilde{M}_{m',n}, \tag{2.10}$$

where we introduced the notation

$$\theta(m \geq n) = \begin{cases} 1, & m \geq n, \\ 0, & n \geq m+1, \end{cases} \tag{2.11}$$

and the ordered product of matrices,

$$\overleftarrow{\prod_{l=n+1}^{m-1}} u_l = \begin{cases} u_{m-1}u_{m-2}\dots u_{n+1}, & m \geq n+2, \\ 1, & m = n+1, \end{cases} . \tag{2.12}$$

Because of Eqs. (1.3), (1.8), and (2.1)

$$u_n = u = w^\sigma, \qquad n \leq k-1, \qquad n \geq K+1, \tag{2.13}$$

i.e., u is a diagonal matrix independent on the regularization parameter λ . Let us consider first $m \leq k$. Then by (2.13) we can rewrite (2.10) in the form

$$u^{-m}\widetilde{M}_{m,n} - \theta(m \geq n+1)u^{-n-1} = u^{-m'}\widetilde{M}_{m',n} - \theta(m' \geq n+1)u^{-n-1}.$$

We see that both sides of this equality are independent either on m, or on m'; we denote them as F_n and thus we get

$$\widetilde{M}_{m,n} = \theta(m \geq n+1)u^{m-n-1} + u^m F_n, \qquad m \leq k. \tag{2.14}$$

Now we chose in (2.10) $m' = k$ and substitute $\widetilde{M}_{k,n}$ in the r.h.s. using (2.14), then

$$\widetilde{M}_{m,n} = \theta(m \geq n+1)\theta(n \geq k) \overleftarrow{\prod_{l=n+1}^{m-1}} u_l + \\ +\theta(k \geq n+1)\left(\overleftarrow{\prod_{l=k}^{m-1}} u_l\right) u^{k-n-1} + \left(\overleftarrow{\prod_{l=k}^{m-1}} u_l\right) u^k F_n,$$

where $m \geq k$. Thus the second term also obeys condition $m \geq n+1$, so that taking (1.3) into account we can write

$$\widetilde{M}_{m,n} = \theta(m \geq n+1) \overleftarrow{\prod_{l=n+1}^{m-1}} u_l + \left(\overleftarrow{\prod_{l=k}^{m-1}} u_l\right) u^k F_n, \qquad m \geq k. \tag{2.15}$$

Eqs. (2.14) and (2.15) give the general solution of (1.17) for any F_n. In order to fix it, we use the two conditions formulated above. First of all it is necessary to guarantee convergency for any n of the series $\sum_m \zeta^{-m} M_{m,n} = \sum_m (h\zeta)^{-m} \widetilde{M}_{m,n}$, where Eq. (2.8) was used. Let us consider first the sum from $-\infty$ to k. Using (2.14) we see that the sum of the first terms is finite due to θ-function. The sum of the second terms in (2.14) is equal (up to a constant factor) to $\sum_{m=-\infty}^{k} u^m (h\zeta)^{-m} F_n$. Thanks to (2.13) this sum converges iff the first (second) row of matrix F_n is equal to zero when $|w| < h$ ($1/|w| < h$, correspondingly). Thus the condition of convergency of this series can be written as

$$\theta(h - |w|^\sigma) F_n = 0, \qquad n \in \mathbb{Z}, \tag{2.16}$$

where the matrix θ-function is defined in (1.25). Let us consider now the condition of convergency of the series $\sum_m \zeta^{-m} M_{m,n}$ at plus infinity. For this purpose we write Eq. (2.15) for $m \geq K+1$ (see (1.3) and (2.13)) as

$$\widetilde{M}_{m,n} = u^{m-1}\left\{\theta(n \geq K)u^{-n} + \theta(K \geq n+1)u^{-K} \overleftarrow{\prod_{l=n+1}^{K}} u_l + \right. \\ \left. +u^{-K}\left(\overleftarrow{\prod_{l=k}^{K}} u_l\right) u^k F_n \right\} - \theta(n \geq m)u^{m-n-1}. \tag{2.17}$$

By (2.8) the series $\sum_{m=K+1}^{\infty}(h\zeta)^{-m}\widetilde{M}_{m,n}$ must be convergent. The sum of the last terms is finite due to the θ-function and the sum of the first terms is convergent iff

$$\theta(|u|-h)\Bigg\{\theta(n\geq K)u^{-n}+\theta(K\geq n+1)u^{-K}\overleftarrow{\prod_{l=n+1}^{K}}u_l+$$

$$+u^{-K}\left(\overleftarrow{\prod_{l=k}^{K}}u_l\right)u^kF_n\Bigg\}=0. \tag{2.18}$$

The conditions (2.16) and (2.18) determine F_n uniquely. In order to get its explicit form, we have to consider the four regions of continuity of the matrices (1.25):

$$h>|w|,\qquad h>1/|w|, \tag{2.19}$$

$$|w|>h>1/|w|, \tag{2.20}$$

$$1/|w|>h>|w|, \tag{2.21}$$

$$|w|>h,\qquad 1/|w|>h. \tag{2.22}$$

Then $\widetilde{M}_{m,n}$ is constructed explicitly using Eqs. (2.14) and (2.15).

Let us introduce the (infinite) matrix column

$$x_m(w,\lambda)=\theta(m\geq k+1)\left(\overleftarrow{\prod_{l=k}^{m-1}}u_l(w,\lambda)\right)w^{k\sigma}+\theta(k\geq m)w^{m\sigma}, \tag{2.23}$$

and row

$$y_n(w,\lambda)=\theta(n\geq K)w^{-n\sigma}+\theta(K\geq n+1)w^{-K\sigma}\overleftarrow{\prod_{l=n+1}^{K}}u_l(w,\lambda). \tag{2.24}$$

In what follows we also use

$$X_m(w,\lambda,h)=h^{-m}x_m(w,\lambda),\qquad Y_n(w,\lambda,h)=h^ny_n(w,\lambda). \tag{2.25}$$

Let a denote a constant (i.e., independent on m and n) matrix

$$a(w,\lambda)=w^{-K\sigma}\overleftarrow{\prod_{l=k}^{K}}u_l(w,\lambda)w^{k\sigma}. \tag{2.26}$$

Then, combining (2.14) and (2.15), we get

$$\widetilde{M}_{m,n}=\theta(m\geq n+1)\overleftarrow{\prod_{l=n+1}^{m-1}}u_l+x_mF_n \tag{2.27}$$

and instead of (2.18) we can write

$$\theta(|w^{\sigma}|-h)aF_n=-\theta(|w^{\sigma}|-h)y_n. \tag{2.28}$$

Let us consider first the region (2.19). Then $\theta(h-|w^{\sigma}|)=1$ and by (2.16) $F_n=0$, so that (2.28) is satisfied identically. This means that in this region

$$\widetilde{M}_{m,n}=\theta(m\geq n+1)\overleftarrow{\prod_{l=n+1}^{m-1}}u_l. \tag{2.29}$$

In the region (2.20)

$$\theta(h-|w^\sigma|)=\frac{1-\sigma}{2}, \qquad \frac{1-\sigma}{2}F_n=0,$$

where Eq. (2.16) was used and by (2.28)

$$F_n=-\frac{1+\sigma}{2a_{1,1}}y_n. \tag{2.30}$$

Analogously in the region (2.21) we have that

$$F_n=-\frac{1-\sigma}{2a_{2,2}}y_n. \tag{2.31}$$

Finally, in the region (2.22) $\theta(h-|w^\sigma|)=0$, thus Eq. (2.16) is satisfied identically and (2.18) takes the form

$$F_n=-a^{-1}y_n. \tag{2.32}$$

All this enables us to write that

$$\widetilde{M}_{m,n}=\theta(m\geq n+1)\overleftarrow{\prod_{l=n+1}^{m-1}}u_l-x_m\Gamma y_n, \tag{2.33}$$

where we introduced the matrix Γ independent on m and n,

$$\Gamma=\begin{cases} 0, & h>|w| \quad \text{and} \quad h>1/|w|, \\ \dfrac{1+\sigma}{2a_{1,1}}, & |w|>h \quad \text{and} \quad h>1/|w|, \\ \dfrac{1-\sigma}{2a_{2,2}}, & h>|w| \quad \text{and} \quad 1/|w|>h, \\ a^{-1}, & |w|>h \quad \text{and} \quad 1/|w|>h. \end{cases} \tag{2.34}$$

Let us mention that thanks to (2.13) we can rewrite Eqs. (2.24) and (2.32) as

$$F_n=-\theta(n\geq k)w^{-k\sigma}\left(\overleftarrow{\prod_{l=k}^{n}}u_l\right)^{-1}-\theta(k\geq n+1)w^{-(n+1)\sigma}, \tag{2.35}$$

and then after some simple calculations we get instead of (2.33)

$$\widetilde{M}_{m,n}=-\theta(n\geq m)\left(\overleftarrow{\prod_{l=m}^{n}}u_l\right)^{-1} \tag{2.36}$$

in the region (2.22).

The results of the above construction can be summarized as the following

Theorem 1. The extended resolvent $M(w,\lambda)$ of the L-operator (1.7) regularized by (2.1) exists, is unique and equals to

$$M_{m,n}(w,\lambda,h)=h^{n-m}\theta(m\geq n+1)\overleftarrow{\prod_{l=n+1}^{m-1}}u_l(w,\lambda)- \\ -X_m(w,\lambda)\Gamma(w,\lambda,h)Y_n(w,\lambda), \tag{2.37}$$

where X_m, Y_n, and Γ are given in (2.23), (2.24), (2.25), and (2.34).

To prove the theorem we need to check first of all that the double series

$$M(w,\lambda,\zeta,\zeta',h)=\sum_{m,n=-\infty}^{+\infty}\zeta^{-m}\zeta'^{n}M_{m,n}(w,\lambda,h), \tag{2.38}$$

where (2.8) was taken into account, converge in the sense of Definition 1. Then it is necessary to prove that (2.37) obeys both (regularized) equations (1.16), i.e., equations (2.5).

The solution (2.37) by construction is the unique solution of the first equation in (2.5) for which the series $\sum_{m=-\infty}^{+\infty}\zeta^{-m}M_{m,n}(w,\lambda,h)$ converge. Convergency of the series in ζ', as well as the second equation, are proved analogously. Both these equations easily follows from (2.37), if we notice that by (2.23) and (2.24)

$$x_{m+1}(w,\lambda)=u_m(w,\lambda)x_m(w,\lambda), \tag{2.39}$$

$$y_{m-1}(w,\lambda)=y_m(w,\lambda)u_m(w,\lambda). \tag{2.40}$$

In other words x_m and y_n are solutions of the equation (1.5) regularized by (2.1) and its dual. By means of (2.25) we can also write these equations as

$$L(w,\lambda)X(w,\lambda)=0, \qquad Y(w,\lambda)L(w,\lambda)=0. \tag{2.41}$$

We see that formally X and Y are right and left annulators of operator L. The existence of these annulators does not contradict (2.5), i.e., the existence of the inversion of L as both series $\sum_m\zeta^{-m}X_m(w,\lambda,h)$ and $\sum_n\zeta^{n}Y_n(w,\lambda,h)$ are divergent, so X_m and Y_n do not belong to the space mentioned in discussion of Eq. (1.9) and in Definition 1. The use of such quantities can be avoided if, say, in the region $|w|>h$, $1/|w|>h$ we use instead of (2.37) the equality

$$M_{m,n}=-h^{n-m}\theta(n\geq m)\left(\overleftarrow{\prod_{l=m}^{n}}u_l(w,\lambda)\right)^{-1}, \tag{2.42}$$

that follows from (2.8) and (2.36).

3 Extended resolvent of the original operator

In order to get the resolvent of the extended original operator (1.7) we need to consider the behavior of (2.37) in the limit $\lambda\to 0$. The existence of this limit depends on the regions (2.19)–(2.22). Indeed, the only origin of a singularity in (2.37) is matrix Γ, as follows from (2.23) and (2.24). Its limits in the first three regions of (2.34) exist by (2.26). Let

$$a(w)=\lim_{\lambda\to 0}a(w,\lambda)=w^{-K\sigma}\left(\overleftarrow{\prod_{l=k}^{K}}u_l(w)\right)w^{k\sigma}. \tag{3.1}$$

This expression is finite and nonzero for generic w. Zeroes of $a_{1,1}(w)$ and $a_{2,2}(w)$ if they exist in the corresponding regions give bound states of operator $L(w)$ and will be studied in the following publication. In the region (2.22) $a^{-1}(w,\lambda)$ has pole at $\lambda=0$, as follows from the last line of (2.34).

To describe the multiplicity of this pole we introduce $q(m,n)$, $m \le n$, number of the degenerated matrices $u_l(w)$ on the interval $[m,n]$, i.e.,

$$q(m,n) = \sum_{l=m}^{n} (1 - \det u_l(w)), \tag{3.2}$$

which is independent on w as $\det u_l(w)$ equals either 0, or 1. Let also

$$Q = q(k,K). \tag{3.3}$$

Then by (2.1) and (2.26) we have that

$$a^{-1}(w,\lambda) = \frac{w^{-k\sigma}}{\lambda^Q} \left(\overrightarrow{\prod_{l=k}^{K}} \left[\lambda w (1 - \det u_l) \frac{1-\sigma}{2} + \tilde{u}_l(w) \right] \right) w^{K\sigma}, \tag{3.4}$$

where we introduce the matrices

$$\tilde{u}_l(w) = \begin{pmatrix} 1/w & -r_n \\ -s_n & w \end{pmatrix} \equiv w^{-\sigma} - \begin{pmatrix} 0 & r_n \\ s_n & 0 \end{pmatrix}, \qquad n \in \mathbb{Z}, \tag{3.5}$$

which are the inverses of $u_n(w)$ in the case where $\det u_n = 1$ (cf. (1.8)). From (3.4) it follows that a^{-1} has pole of order Q at $\lambda = 0$ and we can write

$$a^{-1}(w,\lambda) = \sum_{j=0}^{Q} \frac{t^{(j)}(w)}{\lambda^j} + O(\lambda), \qquad \lambda \to 0. \tag{3.6}$$

The residues are equal to

$$t^{(j)}(w) = w^{Q-k\sigma} \left(\overrightarrow{\prod_{l=k}^{K}} \tilde{u}_l(w) \right)^{(Q-j)} w^{K\sigma}, \tag{3.7}$$

where by definition

$$\begin{aligned} \left(\overrightarrow{\prod_{l=m}^{n}} \tilde{u}_l(w) \right)^{(j)} &= \sum_{m \le l_1 < \ldots < l_j \le n} \left(\prod_{i=1}^{j} (1 - \det u_{l_i}(w)) \right) \times \\ &\times \left[\overrightarrow{\prod_{l=m}^{n}} \tilde{u}_l(w) \right]_{\substack{\tilde{u}_{l_i} = \frac{1-\sigma}{2}, \\ i=1,\ldots,j}} \end{aligned} \tag{3.8}$$

for any $m \le n$.

By (2.23), (2.25), and (2.24) $X_m(w,\lambda)$ and $Y_n(w,\lambda)$ are polynomials in λ, so that we have the Laurent expansion

$$M(w,\lambda,h) = \widehat{M}(w,\lambda,h) + \sum_{j=1}^{Q} \frac{M^{(j)}(w)}{\lambda^j}, \tag{3.9}$$
$$|w| > h \quad \text{and} \quad 1/|w| > h,$$

where $\widehat{M}$ is the regular part of the series. The residues can be calculated explicitly by Eqs. (2.37) and (3.6), but in order to work with objects belonging to the space mentioned in Definition 1 it is reasonable to use the representation (2.42). Then by means of notations (3.2) and (3.8) we get

$$M^{(j)}_{m,n}(w,h) = -h^{n-m}\theta(n \geq m)\theta(q(m,n) \geq j) \times \\ \times w^{q(m,n)-j}\left(\prod_{l=m}^{\overrightarrow{n}} \tilde{u}_l(w)\right)^{(q(m,n)-j)}. \tag{3.10}$$

Now the resolvent $M(w,h)$ of the original (extended) L-operator (1.7) can be defined as

$$M(w,h) = \begin{cases} M(w,\lambda,h)\mid_{\lambda=0}, & h > |w| \quad \text{or} \quad h > 1/|w|, \\ \widehat{M}(w,\lambda,h)\mid_{\lambda=0}, & |w| > h \quad \text{and} \quad 1/|w| > h. \end{cases} \tag{3.11}$$

Let us consider region $|w| > h$, $1/|w| > h$ in detail. Inserting the expression (3.9) into Eqs. (2.6) and (2.7) and using (3.11) we derive that

$$L(w)M(w) = I + DM^{(1)}(w), \quad M(w)L(w) = I + M^{(1)}(w)D, \tag{3.12}$$
$$L(w)M^{(j)}(w) = I + DM^{(j+1)}(w), \tag{3.13}$$
$$M^{(j)}(w)L(w) = I + M^{(j+1)}(w)D, \qquad j = 1,\dots,Q-1, \tag{3.14}$$
$$L(w)M^{(Q)}(w) = 0, \quad M^{(Q)}(w)L(w) = 0. \tag{3.15}$$

Thus we see that in this region equations (3.12) defining the resolvent are modified in comparison with the standard Eqs. (1.16). In [26] it was shown that a solution of (1.16) does not exist in this region. In order to study the properties of the residues $M^{(j)}$ we can use Hilbert identity

$$M(w,\lambda')[L(w,\lambda') - L(w,\lambda)]M(w,\lambda) = M(w,\lambda) - M(w,\lambda')$$

that follows from (2.5). Taking into account (2.6) and (2.7) we can rewrite it in the form

$$M(w,\lambda) - M(w,\lambda') = (\lambda' - \lambda)M(w,\lambda')DM(w,\lambda). \tag{3.16}$$

Substituting $M(w,\lambda')$ as in Eq. (3.9), we get in the limit $\lambda' \to 0$ that

$$M(w,\lambda) - M(w) = [\lambda M(w) - M^{(1)}(w)]DM(w,\lambda), \tag{3.17}$$
$$M^{(j)}(w) = [M^{(j+1)}(w) - \lambda M^{(j)}(w)]DM(w,\lambda), \tag{3.18}$$

where $j = 1,\dots,Q$, and we put by definition

$$M^{(Q+1)}(w) \equiv 0. \tag{3.19}$$

Now we insert the expansion (3.11) in the Eqs. (3.17) and (3.18), and by (3.17) in the limit $\lambda \to 0$ we derive that

$$M^{(1)}(w)DM(w) = M(w)DM^{(1)}(w), \tag{3.20}$$
$$M^{(j)}(w) = M(w)DM^{(j+1)}(w) - M^{(1)}(w)DM^{(j)}(w), \tag{3.21}$$
$$j = 1,\dots,Q.$$

Then from (3.18) we have

$$M^{(j)}(w) = M^{(j+1)}(w)DM(w) - M^{(j)}(w)DM^{(1)}(w) \tag{3.22}$$

(symmetric to (3.21)) and

$$M^{(j+1)}(w)DM^{(l)}(w) = M^{(j)}(w)DM^{(l+1)}(w), \tag{3.23}$$

$$M^{(j)}(w)D\widehat{M}(w,\lambda) = \frac{1}{\lambda}M^{(j+1)}(w)D[\widehat{M}(w,\lambda) - M(w)], \tag{3.24}$$

$$j,l = 1,\dots,Q,$$

where (3.11) and (3.19) were used. By (3.22) we have $M^{(Q)}(w) = -M^{(Q)}(w)DM^{(1)}(w)$ and by (3.24) $M^{(Q)}(w)D\widehat{M}(w,\lambda) = 0$, so that also $M^{(Q)}(w)DM(w) = 0$ by (3.11). Then for $j = Q-1$ in (3.22) we have that $M^{(Q-1)}(w) = -M^{(Q-1)}(w)DM^{(1)}(w)$ and from (3.24) that
$M^{(Q-1)}(w)D\widehat{M}(w,\lambda) = 0$, so that again $M^{(Q-1)}(w)DM(w) = 0$. Continuing in this way we prove that

$$M^{(j)}(w) = -M^{(j)}(w)DM^{(1)}(w), \tag{3.25}$$

$$M^{(j)}(w) = -M^{(1)}(w)DM^{(j)}(w), \tag{3.26}$$

and

$$M^{(j)}(w)DM(w) = 0, \quad M(w)DM^{(j)}(w) = 0, \tag{3.27}$$

$$j = 1,\dots,Q,$$

where (3.26) and the second set of equations in (3.27) is derived in an analogous way from (3.16), if λ and λ' are interchanged.

Now by (3.23) we get

$$M^{(j)}(w)DM^{(l)}(w) = M^{(j-k)}(w)DM^{(l+k)}(w), \tag{3.28}$$

where $j,l = 1,\dots,Q$, $k \geq 0$, $j-k \geq 1$, $l+k \leq Q+1$. If j and l are such that $j-k \geq 1$ and $l+k = Q+1$, i.e., $l+j \geq Q+2$, then the r.h.s. of (3.28) is equal to zero thanks to (3.19). On the other side if $l+j \leq Q+1$ we can chose in (3.28) $k = j-1$ and then by (3.25) or (3.26) we get finally

$$M^{(j)}(w)DM^{(l)}(w) = -\theta(Q+1 \geq l+j)M^{(l+j-1)}(w), \tag{3.29}$$

$$j,l = 1,\dots,Q,$$

that replaces relations (3.23), (3.25), and (3.26).

This concludes the construction of the resolvent $M_{m,n}(w,h)$ of the extended L-operator (1.7). As we have seen, this resolvent is discontinuous at $|w| = h$ and $|w| = 1/h$. In a forthcoming paper we show that study of this discontinuity leads us to modification of the Jost solutions and spectral data, corresponding to the case of the discrete potential (1.2).

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